TRIGONOMETRY

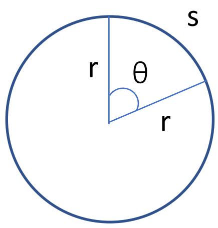
ANGLE MEASUREMENT

π radius = 180°

$1^{\circ} = \frac{\pi}{180}\text{rad} \quad 1\text{ rad} = \frac{180^{\circ}}{\pi}$

$s = r\,\theta$

(θ in radians)



FUNDMENTAL IDENTITIES

$\csc\theta = \frac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$ $\qquad$ $sin^2\,\theta + cos^2\theta = 1$

$1 + tan^2\,\theta = \sec^2\theta$ $\qquad$ $1 + cot^2\,\theta = \csc^2\theta$

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\sin(\frac{\pi}{2} - \theta) = \cos\theta$

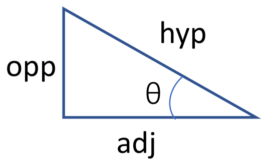
$\cos(\frac{\pi}{2} - \theta) = \sin\theta$ $\qquad$ $\tan(\frac{\pi}{2} - \theta) = \cot\theta$

RIGHT ANGLE TRIGONOMETRY

$\sin\theta = \frac{\text{opp}}{hyp}$ $\qquad$ $\csc\theta = \frac{hyp}{\text{opp}}$

$\cos\theta = \frac{adj}{\text{hyp}}$ $\qquad$ $\sec\theta = \frac{hyp}{adj}$

$\tan\theta = \frac{opp}{\text{adj}}$ $\qquad$ $\csc\theta = \frac{adj}{\text{opp}}$

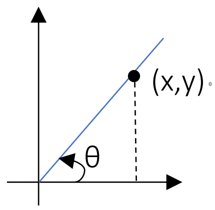


TRIGONOMETRIC FUNCTIONS

$\sin\theta = \frac{\text{y}}{r}$ $\qquad$ $\csc\theta = \frac{\text{r}}{y}$

$\cos\theta = \frac{\text{x}}{r}$ $\qquad$ $\sec\theta = \frac{\text{r}}{x}$

$\tan\theta = \frac{\text{y}}{x}$ $\qquad$ $\cot\theta = \frac{\text{x}}{y}$



ADDITION AND SUBTRACTION FORMULAS

$\sin(x+y) = sinx\ cosy + cosx\ siny$

$\sin(x-y) = sinx\ cosy - cosx\ siny$

$\cos(x+y) = cosx\ cosy - sinx\ siny$

$\cos(x-y) = cosx\ cosy + sinx\ siny$

DOUBLE-ANGLE FORMULAS

$\sin 2x = 2\ sinx\ cosx$

$\cos 2x = cos^2 x - sinx^2$

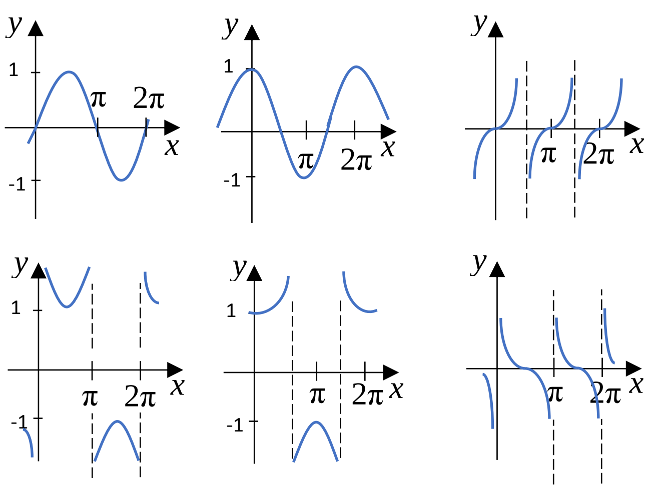
$\qquad = 2cosx^2 - 1 = 1 - 2sin^2 x$

$\tan 2x = \frac{2\tan x}{1 - tan^2\,x}$

HALF-ANGLE FORMULAS

$sin^2\,x = \frac{1-\cos 2x}{2}$ $\qquad$ $cos^2\,x = \frac{1+\cos 2x}{2}$

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS



TRIGONOMETRIC FUNCTIONS OF IMPORTANT ANGLES

θ	radius	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	$\pi/2$	1	0	--

EXAMPLE 6 **(Too many wiggles)** Find $\lim_{x \to 0} \sin(1/x)$.

SOLUTION This example poses the most subtle limit question asked yet. Since we do not want to make too big a story out of it, we ask you to do two things. First, pick a sequence of x-values approaching 0. Use your calculator to evaluate $\sin(1/x)$ at these x's. Unless you happen on some very lucky choices, your values will oscillate wildly.

Second, consider trying to graph $y = \sin(1/x)$. No one will ever do this very well, but the table of values in Figure 8 gives a good clue about what is happening. In any neighborhood of the origin, the graph wiggles up and down between -1 and 1 infinitely many times (Figure 9). Clearly, $\sin(1/x)$ is not near a single number L when x is near 0. We conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. ■

x	$\sin\frac{1}{x}$
$2/\pi$	1
$2/(2\pi)$	0
$2/(3\pi)$	-1
$2/(4\pi)$	0
$2/(5\pi)$	1
$2/(6\pi)$	0
$2/(7\pi)$	-1
$2/(8\pi)$	0
$2/(9\pi)$	1
$2/(10\pi)$	0
$2/(11\pi)$	-1
$2/(12\pi)$	0
$\downarrow$	$\downarrow$
0	?

Figure 8

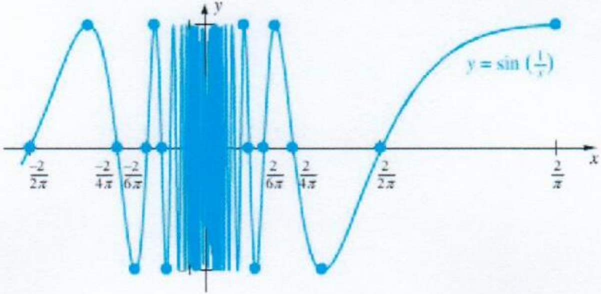


Figure 9

△ Example 4. Show that $\lim_{x \to 0} \sin\frac{1}{x}$ doesn't exist

$\sin\frac{1}{x}$ $^{\cos\frac{1}{x}}$ $\sin\frac{\pi}{x}$, $\cos\frac{\pi}{x}$

<proof>. See Figure 9. (p.74)

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$

(1) Consider $f(x) = 0 \Rightarrow \frac{1}{x} = n\pi$, where n is an integer $= 0 \Rightarrow ??$ $\neq 0$

$\Rightarrow x = \frac{1}{n\pi}$

Take a sequence $\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{n\pi}\right\}_{n=1}^{\infty} = \left\{\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \cdots\right\}$

$\Rightarrow f(x_n) = \sin n\pi = 0$ as $x_n \to 0^+$

(2) Consider $f(x) = 1 \Rightarrow \frac{1}{x} = \frac{\pi}{2} + 2n\pi$, where n is an integer

$\Rightarrow x = \frac{1}{\frac{\pi}{2} + 2n\pi}$

Take a sequence $\{x_n\} = \left\{\frac{1}{\frac{\pi}{2}}, \frac{1}{\frac{\pi}{2}+2\pi}, \frac{1}{\frac{\pi}{2}+3\pi}, \cdots\right\}$

$\Rightarrow f(x_n) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$ as $x_n \to 0^+$

Hence, by (1) (2),

$\Rightarrow \lim_{x \to 0} \sin\frac{1}{x}$ doesn't exist.

Intermediate value theorem (IVT) 中間值定理

If f is continuous on $[a,b]$, and let w be a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there is at least one number $c \in (a,b)$ such that $f(c) = w$.

Example 3.

Use the IVT to show that the equation $x - \cos x = 0$ has a solution between $x=0$ and $x=\frac{\pi}{2}$.

(proof) Let $f(x) = x - \cos x$.

$\because$ f is continuous on $[0,\frac{\pi}{2}]$ everywhere and

$f(0) = 0 - \cos 0 = -1 < 0$, $f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$

w

By IVT

$\therefore$ there is at least one $\in (0,\frac{\pi}{2})$

such that $f(c) = 0$ w

$\Rightarrow c - \cos c = 0$